

# FOURIER TRANSFORM FOR QUANTUM $D$ -MODULES VIA THE PUNCTURED TORUS MAPPING CLASS GROUP

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**ABSTRACT.** We construct a certain cross product of two copies of the braided dual  $\tilde{H}$  of a quasitriangular Hopf algebra  $H$ , which we call the elliptic double  $E_H$ , and which we use to construct representations of the punctured elliptic braid group extending the well-known representations of the planar braid group attached to  $H$ . We show that the elliptic double is the universal source of such representations. We recover the representations of the punctured torus braid group obtained in [Jo], and hence construct a homomorphism to the Heisenberg double  $D_H$ , which is an isomorphism if  $H$  is factorizable.

The universal property of  $E_H$  endows it with an action by algebra automorphisms of the mapping class group  $\widetilde{SL_2(\mathbb{Z})}$  of the punctured torus. One such automorphism we call the quantum Fourier transform; we show that when  $H = U_q(\mathfrak{g})$ , the quantum Fourier transform degenerates to the classical Fourier transform on  $D(\mathfrak{g})$  as  $q \rightarrow 1$ .

## 1. INTRODUCTION

Let  $(H, \mathcal{R})$  be a quasi-triangular Hopf algebra, and let  $\tilde{H}$  denote the braided dual – also known as the reflection equation algebra – of  $H$  [DKM, DM2, DM1, Ma]. This is the restricted dual vector space  $H^\circ$ , but the multiplication is twisted from the standard one by the  $R$ -matrix (see Section 2 for details).

Let  $\{e_i\}$  and  $\{e^i\}$  denote dual bases of  $H$  and  $\tilde{H}$ , respectively. Then the canonical element  $X = \sum e^i \otimes e_i \in \tilde{H} \otimes H$  is known to satisfy the following relation in  $\tilde{H} \otimes H^{\otimes 2}$ :

$$X^{0,12} := (\text{id} \otimes \Delta)(X) = (\mathcal{R}^{1,2})^{-1} X^{0,2} \mathcal{R}^{1,2} X^{0,1} \quad (1.1)$$

Here,  $\tilde{H}$  has index 0 in the tensor product, and  $\Delta$  denotes the coproduct of  $H$ .

There is a canonical action of the planar braid group  $B_n(\mathbb{R}^2)$  on the  $n$ th tensor  $V^{\otimes n}$  power of any  $H$ -module  $V$ . Given modules  $M$  for  $\tilde{H}$  and  $V$  for  $H$ , equation (1.1) allows one to define a similarly canonical action of the punctured planar braid group  $B_n(\mathbb{R}^2 \setminus \text{disc})$  on  $M \otimes V^{\otimes n}$ , and moreover to show that  $\tilde{H}$  is universal for this action. We have:

**Theorem 1.1** ([DKM], Prop 10). *Let  $B$  be an algebra, and suppose that  $X_B \in B \otimes H$  satisfies relation (1.1). Then there is a unique homomorphism  $\phi_B : \tilde{H} \rightarrow B$  such that  $(\phi_B \otimes \text{id})(X) = X_B$ .*

The main goal of this paper is to define elliptic analogs of the reflection equation algebra. The punctured elliptic braid group  $B_n(T^2 \setminus \text{disc})$  is the free product of two copies of  $B_n(\mathbb{R}^2 \setminus \text{disc})$ , modulo certain relations. In Section 3 we construct an algebra  $E_H$  as a certain crossed product of two copies of  $\tilde{H}$ , mimicking the cross relations of  $B_n(T^2 \setminus \text{disc})$ . We define canonical elements  $X, Y \in E_H \otimes H$  by

$$X = \sum (e^i \otimes 1) \otimes e_i, \quad Y = \sum (1 \otimes e^i) \otimes e_i,$$

and characterize the cross relations on  $E_H$  as follows:

**Theorem 1.2.** *The cross relations of  $E_H$  are equivalent to the following commutation relation for  $X, Y, \mathcal{R}$ :*

$$X^{0,1}\mathcal{R}^{2,1}Y^{0,2} = \mathcal{R}^{2,1}Y^{0,2}\mathcal{R}^{1,2}X^{0,1}\mathcal{R}^{2,1} \quad (1.2)$$

We prove the following elliptic analog of Theorem 1.1:

**Theorem 1.3.** *Let  $B$  be an algebra, and  $X_B, Y_B \in B \otimes H$  satisfying (1.1) individually, and (1.2) together. Then there exists a unique algebra morphism*

$$\phi_B : E_H \longrightarrow B$$

*such that  $X_B = (\phi_B \otimes \text{id})(X)$  and  $Y_B = (\phi_B \otimes \text{id})(Y)$ . Explicitly,  $\phi_B$  is given by*

$$\phi_B(x \otimes 1) = (\text{id} \otimes x)(X_B) \quad \phi_B(1 \otimes x) = (\text{id} \otimes x)(Y_B).$$

Equation (1.2) can be used to define representations of  $B_n(T^2 \setminus \text{disc})$  in the same way as (1.1) is used for  $B_n(\mathbb{R}^2 \setminus \text{disc})$ ; see Theorem 4.3. Recall that  $B_n(T^2 \setminus \text{disc})$  carries a natural action of the punctured torus mapping class group, which is isomorphic to a certain central extension  $\widetilde{SL_2(\mathbb{Z})}$  of  $SL_2(\mathbb{Z})$ . In the case  $H$  is a ribbon Hopf algebra, we show that this extends to an action of  $\widetilde{SL_2(\mathbb{Z})}$  on  $E_H$ .

When  $H = U_q(\mathfrak{g})$ , we produce degenerations of  $E_H$  to the algebras of differential operators on  $G$  and, upon further degeneration, on  $\mathfrak{g}$ . Recall that the algebra of differential operators on an algebraic group  $G$  can be constructed as a semi-direct product

$$D(G) = U(\mathfrak{g}) \ltimes O(G),$$

where the action of  $U(\mathfrak{g})$  on  $O(G)$  is induced by that of  $\mathfrak{g}$  on  $G$  by left invariant differential operators. This construction can be extended to any Hopf algebra and is known as the Heisenberg double [STS]. This is a semi-direct product  $D_H = H \ltimes H^\circ$ , where  $H$  acts on its dual by the right coregular action.

In [Jo], canonical functors are constructed from the category of modules over the Heisenberg double of a quasi-triangular Hopf algebra to the category of modules over the (unpunctured) torus braid group. This relies upon an alternate construction – due to Varagnolo-Vasserot [VV] – of the Heisenberg double of a quasi-triangular Hopf algebra, which uses the braided dual  $\check{H}$  in place of  $H^\circ$ . This presentation for the Heisenberg double also yields an isomorphism with the handle algebras  $S_{1,1}$  of [AGS] (see Remark 3.4).

Lifting the constructions of [Jo] to the unpunctured torus braid group, they can easily be re-interpreted as producing canonical elements  $X$  and  $Y$  in  $D_H \otimes H$ , satisfying equations (1.1) and (1.2). Hence, Theorem 1.3 yields a unique homomorphism  $\Phi : E_H \rightarrow D_H$ , compatible with the representations of the  $B_n(T^2 \setminus \text{disc})$  on both sides. The map  $\Phi$  is an isomorphism if, and only if,  $H$  is *factorizable*. Since the quantum group  $U_q(\mathfrak{g})$  is factorizable, we may identify the elliptic double  $E_{U_q(\mathfrak{g})}$  with the algebra  $D_q(G) := D_{U_q(\mathfrak{g})}$  of quantum differential operators on  $G$ .

In particular we obtain an  $\widetilde{SL_2(\mathbb{Z})}$  action on  $D_q(G)$  by the above considerations. One such automorphism of  $D_q(G)$  we call the *quantum Fourier transform*; its classical limit upon an appropriate degeneration is the classical Fourier transform on the Weyl algebra  $D(\mathfrak{g})$ . We expect that our quantum Fourier transform for  $D_q(G)$  will be compatible with that on the braided dual of  $U_q(\mathfrak{g})$  defined in [LM], realizing the braided dual as an  $\widetilde{SL_2(\mathbb{Z})}$ -equivariant  $D_q(G)$ -module. Studying this category of  $\widetilde{SL_2(\mathbb{Z})}$ -equivariant  $D_q(G)$ -modules more generally is an interesting direction of future research.

**Acknowledgments.** This paper is a companion to work in progress with D. Ben-Zvi [BZBJ], in which we generalize the elliptic double construction to arbitrary genus, and to any braided tensor category, using the language of topological field theory. We are grateful to D. Ben-Zvi, and to all three authors of [CEE], for their many helpful discussions and encouragement, and to P. Roche for bringing the article [AGS] to our attention.

## 2. THE BRAIDED DUAL AND ITS RELATIVES

Let  $(H, \mathcal{R})$  be a quasi-triangular Hopf algebra, and denote by:

- $H^e = H^{coop} \otimes H$  where  $H^{coop}$  is  $H$  with opposite comultiplication
- $H^{[2]}$  the Hopf algebra which is  $H \otimes H$  as an algebra, and with coproduct given by

$$\tilde{\Delta}(x \otimes y) = (\mathcal{R}^{2,3})^{-1}(\tau^{2,3} \circ \Delta(x \otimes y))\mathcal{R}^{2,3}$$

where  $\tau(a \otimes b) = b \otimes a$ . Recall that the twist  $H^F$  of  $H$  by an invertible element  $F \in H \otimes H$  is the Hopf algebra with the same multiplication, and with coproduct given by

$$\Delta^F(x) = F^{-1}\Delta(x)F.$$

In order for  $H^F$  to be co-associative,  $F$  must satisfy two conditions:

$$F^{12,3}F^{1,2} = F^{1,23}F^{2,3}, \quad (\epsilon \otimes \text{id})(F) = (\text{id} \otimes \epsilon)(F) = 1.$$

Two twists  $F, F'$  are *equivalent* if there exists an invertible element  $x \in H$ , such that  $\epsilon(x) = 1$  and

$$F' = \Delta(x)F(x^{-1} \otimes x^{-1}).$$

The following is standard (see [Dr2]):

**Proposition 2.1.** *A twist induces a tensor equivalence  $H\text{-mod} \rightarrow H^F\text{-mod}$ . Equivalent twists leads to isomorphic tensor functors.*

It is easily checked that  $F = \mathcal{R}^{1,3}\mathcal{R}^{1,4} \in (H^e)^{\otimes 2}$  is a twist, and that

$$H^{[2],coop} = (H^e)^F.$$

Let  $D$  be the “double braiding”  $\mathcal{R}^{2,1}\mathcal{R}^{1,2}$ . Since  $D\Delta(x) = \Delta(x)D$  for all  $x$ , we have:

$$H^D = H$$

as Hopf algebras. Similarly,  $H^{[2],coop}$  is in fact equal to  $(H^e)^{F(D^{1,3})^k}$  for any  $k \in \mathbb{Z}$ , with  $F$  as above.

Let  $H^\circ$  be the restricted Hopf algebra dual of  $H$ . It has a natural  $H$ -bimodule structure, hence a  $H^e$  left module structure given by:

$$(x \otimes y) \triangleright f := f(S^{-1}(x) \cdot y)$$

where  $S$  is the antipode of  $H$  and we use the fact that  $S^{-1}$  is a Hopf algebra isomorphism  $H^{coop} \rightarrow H_{op}$ . It turns  $H^\circ$  into an algebra in  $H^e\text{-mod}$ .

*Remark 2.2.* We use the inverse of the antipode rather than the antipode itself because it is convenient to consider the canonical element as an invariant element of  $H^\circ \otimes H$ , the image of  $1 \in \mathbb{C}$  under the evaluation map  $\mathbf{k} \rightarrow H^\circ \otimes H$ , which means that  $H^\circ$  really denotes the *left* dual of  $H$  in the rigid monoidal category of  $H$ -modules. This is slightly different from the convention used in [DKM, Jo] but it allows us to label tensor factors from left to right.

**Definition 2.3.** The  $k$ th twisted braided dual  $\tilde{H}_k$  is the algebra image of  $H^\circ$  via the tensor functor  $H^e\text{-mod} \rightarrow H^{[2],\text{coop}}\text{-mod}$  given by the twist  $F(D^{1,3})^k$ . Explicitly, this is  $H^\circ$  as a vector space, with multiplication given by

$$x \cdot y = m(\mathcal{R}^{1,3}\mathcal{R}^{1,4}(D^{1,3})^k \triangleright (x \otimes y))$$

where  $m$  is the multiplication of  $H^\circ$ . This is an algebra in the category of  $H^{[2],\text{coop}}$ -module with the same action as above, namely

$$(x \otimes y) \triangleright f = (u \mapsto f(S^{-1}(x)uy)).$$

Let  $X$  be the canonical element of  $\tilde{H} \otimes H$ , that is the image of 1 under the coevaluation map  $\mathbf{k} \rightarrow \tilde{H} \otimes H$ . If  $e_i$  is a basis of  $H$  and  $e^i$  the dual basis of  $\tilde{H} \cong H^\circ$ , then  $X = \sum e^i \otimes e_i$ . If  $H$  is infinite dimensional then  $X$  lives in an appropriate completion of the tensor product.

**Proposition 2.4.** The element  $X$  satisfies:

$$X^{0,12} = D^k(\mathcal{R}^{1,2})^{-1} X^{0,2} \mathcal{R}^{1,2} X^{0,1}. \quad (2.1)$$

This implies that  $X$  satisfies the reflection equation

$$\mathcal{R}^{2,1} X^{0,2} \mathcal{R}^{1,2} X^{0,1} = X^{0,1} \mathcal{R}^{2,1} X^{0,2} \mathcal{R}^{1,2}.$$

The braided dual is in fact universal for this property in the following sense:

**Proposition 2.5.** Let  $B$  be an algebra and  $X_B \in B \otimes H$  satisfying equation (2.1) for some  $k \in \mathbb{Z}$ . Then there exists a unique algebra morphism

$$\phi_B : \tilde{H}_k \longrightarrow B$$

such that  $(\phi_B \otimes \text{id})(X) = X_B$ . Explicitly,  $\phi_B$  is given by

$$H^\circ \cong \tilde{H} \ni f \longmapsto (f \otimes \text{id})(X).$$

Propositions 2.4 and 2.5 are proved in [DKM] in the case  $k = 0$ . The general proof is similar. Note that the fact that these axioms all leads to the same reflection equation, regardless of the value of  $k$ , essentially follows from the fact that the left hand side of (2.1) is invariant under conjugation by  $D$ .

Let  $u = m((S \otimes \text{id})(R^{2,1}))$  where  $m$  is the multiplication of  $H$ . Then  $\nu = uS(u)$  is central and satisfies

$$\Delta(\nu) = D^{-2}(\nu \otimes \nu)$$

implying that

$$D^{k-2} = \Delta(\nu) D^k (\nu^{-1} \otimes \nu^{-1})$$

meaning that  $D^{k-2}$  and  $D^k$  are equivalent. Therefore, they lead to isomorphic tensor functors, from which follows the following:

**Proposition 2.6.** For any  $k \in \mathbb{Z}$ , the algebras  $\tilde{H}_k$  and  $\tilde{H}_{k+2}$  are isomorphic.

Therefore, it is enough to consider  $\tilde{H}_0$  and  $\tilde{H}_1$ . Moreover, if  $H$  is a ribbon Hopf algebra, then by definition  $\nu$  admits a central square root implying by a similar argument:

**Proposition 2.7.** If  $H$  is a ribbon Hopf algebra then all the  $\tilde{H}_k$  are isomorphic.

*Remark 2.8.* The algebra  $\tilde{H}_0$  is usually called the reflection dual, the braided dual or the reflection equation algebra in the literature.

*Remark 2.9.* For any  $k$ , equation (2.1) plays the same role in the reflection equation, as the hexagon axiom in the Yang-Baxter equation, encoding some kind of compatibility with the tensor product of  $H$ -modules. Topologically, it corresponds to a “strand doubling” operation for the additional generator of the braid group of the punctured plane. Formally, such an operation depends on the choice of a framing, while a ribbon element removes the dependence on the framing.

## 3. THE ELLIPTIC DOUBLE

Let  $T$  denote the following element in  $(H^{[2],coop})^{\otimes 2}$ , which we identify as a vector space with  $H^{\otimes 4}$ :

$$T = (\mathcal{R}^{3,2})^{-1}(\mathcal{R}^{3,1})^{-1}(\mathcal{R}^{4,2})^{-1}\mathcal{R}^{1,4}.$$

**Proposition 3.1.** *The element  $T$  satisfies the hexagon axioms*

$$(\text{id} \otimes \Delta_{H^{[2],coop}})T = T^{1,3}T^{1,2} \quad (\Delta_{H^{[2],coop}} \otimes \text{id})T = T^{1,3}T^{2,3}$$

in  $(H^{[2],coop})^{\otimes 3}$ .

*Proof.* This is straightforward computation with the Yang-Baxter equation. The computation is depicted in braids in Figure 1.

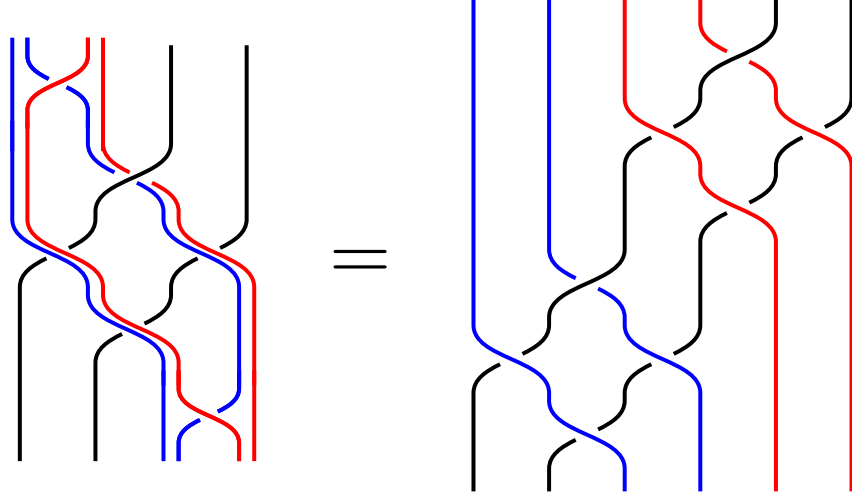


FIGURE 1. A braid diagram proof of  $(\text{id} \otimes \Delta)(T) = T_{1,3}T_{1,2}$ .

□

Since  $\tilde{H}_k$  is a  $H^{[2],coop}$ -module algebra, one can make the following definition:

**Definition 3.2.** *The  $k$ th elliptic double  $E_H^{(k)}$  of  $H$  is the braided tensor square of  $\tilde{H}_k$  with respect to  $T$ . Explicitly, it is  $\tilde{H}_k^{\otimes 2}$  as a vector space,  $\tilde{H}_k \otimes 1$  and  $1 \otimes \tilde{H}_k$  are subalgebras and the cross relations are given by*

$$(1 \otimes g)(f \otimes 1) = T \triangleright (f \otimes g).$$

The fact that  $E_H^{(k)}$  is indeed an associative algebra follows from the hexagon axioms. Choose a basis  $(e_i)_{i \in I}$  of  $H$  and define  $X, Y \in E_H^{(k)} \otimes H$  by

$$X = \sum e^i \otimes 1 \otimes e_i, \quad Y = \sum 1 \otimes e^i \otimes e_i,$$

where we use the vector space identification  $E_H^{(k)} \cong \tilde{H}^{\otimes 2}$ . The main result of this section is the following:

**Theorem 3.3.** *The cross relations of  $E_H$  are equivalent to the commutation relation for  $X, Y, \mathcal{R}$ :*

$$X^{0,1}\mathcal{R}^{2,1}Y^{0,2} = \mathcal{R}^{2,1}Y^{0,2}\mathcal{R}^{1,2}X^{0,1}\mathcal{R}^{2,1}. \quad (3.1)$$

*Proof.* By definition every element  $f \in \tilde{H}_k$  can be written as

$$f = \sum e^i f(e_i)$$

hence the product  $gf$  in  $E_H^{(k)}$  is obtained by applying  $(\text{id}_{E_H^{(k)}} \otimes f \otimes g)$  to

$$Y^{0,2} X^{0,1}$$

and  $fg$  by applying the same element to

$$X^{0,1} Y^{0,2}.$$

Therefore all commutations relation can be gathered into a “matrix” equation

$$Y^{0,2} X^{0,1} = T \triangleright_0 X^{0,1} Y^{0,2} \quad (3.2)$$

where  $T$  acts on the  $E_H^{(k)}$  (i.e. 0th) component. We recall the following identities:

$$\mathcal{R}^{-1} = (S \otimes \text{id})(\mathcal{R}) = (\text{id} \otimes S^{-1})(\mathcal{R}). \quad (3.3)$$

Applying  $S^{-1}$  to the first factor of the relation  $(S \otimes \text{id})(R)R = 1$ , setting  $\mathcal{R} = \sum r_1 \otimes r_2 = \sum r'_1 \otimes r'_2$  – using apostrophes to distinguish between copies of  $\mathcal{R}$  – one has the following useful identity (note the order of the terms):

$$\sum S^{-1}(r_1) r'_1 \otimes r'_2 r_2 = 1. \quad (3.4)$$

Then equation (3.2) reads, in coordinates:

$$\begin{aligned} & ((1 \otimes e^j)(e^i \otimes 1)) \otimes e_i \otimes e_j \\ &= ((r_2 r'_1 \otimes r_2''' r_2'' \otimes S(r_1'''' ) S(r_1'') r'_2) \triangleright e^i \otimes e^j) \otimes e_i \otimes e_j. \end{aligned} \quad (3.5)$$

The left  $H^{[2]}$  action on  $\tilde{H}_k$  is by definition dual to the right  $H^{[2]}$  action on  $H$ , therefore:

$$\sum ((x \otimes y) \triangleright e^i) \otimes e_i = \sum e^i \otimes S^{-1}(x) e_i y$$

Using this, equation (3.5) can be rewritten

$$((1 \otimes e^j)(e^i \otimes 1)) \otimes e_i \otimes e_j = e^i \otimes e^j \otimes S^{-1}(r_1') S^{-1}(r_2) e_i r_2''' r_2'' \otimes r_1 r_1''' e_j S(r_1'') r_2'.$$

Then, using the  $R$ -matrix relations (3.3) and (3.4) to move elements from the right hand side to the left hand side (and reassigning apostrophes for the sake of clarity) we obtain:

$$((1 \otimes e^j)(e^i \otimes 1)) \otimes r_2 r'_1 e_i r_2'' \otimes r_1 e_j r'_2 r_1'' = e^i \otimes e^j \otimes e_i r_2 \otimes r_1 e_j$$

which is exactly (1.2).  $\square$

*Remark 3.4.* The relations of Theorem 3.3 should be compared with those of the graph algebra  $S_{1,1}$  of [AGS].

Equation (1.2) is a defining relation for  $E_H^{(k)}$ , in the following sense:

**Corollary 3.5.** *Let  $B$  be an algebra, and  $X_B, Y_B \in B \otimes H$  satisfying both the axiom (2.1) and equation (1.2) (with  $X$  and  $Y$  replaced by  $X_B$  and  $Y_B$ ). Then there exists a unique algebra morphism*

$$\phi_B : E_H^{(k)} \longrightarrow B$$

*such that  $X_B = (\phi_B \otimes \text{id})(X)$  and  $Y_B = (\phi_B \otimes \text{id})(Y)$ . Explicitly,  $\phi_B$  is given by*

$$\phi_B(x \otimes 1) = (\text{id} \otimes x)(X_B) \quad \phi_B(1 \otimes x) = (\text{id} \otimes x)(Y_B).$$

## 4. BRAID GROUP AND MAPPING CLASS GROUP ACTIONS

In this section we construct representations of the punctured torus braid group from  $E_H^{(k)}$ . First, we have:

**Definition 4.1.** *The punctured elliptic braid group  $B_n(T^2 \setminus \text{disc})$  is the fundamental group of the configuration space of  $n$  points in  $T^2 \setminus \text{disc}$ .*

**Proposition 4.2.** *The group  $B_n(T^2 \setminus \text{disc})$  is generated by  $X_1, \dots, X_n, Y_1, \dots, Y_n, \sigma_1, \dots, \sigma_{n-1}$  with relations:*

- the  $X_i$ 's (resp.  $Y_i$ 's) pairwise commute,
- the planar braid relation for the  $\sigma_i$ 's,
- the following cross relations:

$$X_{i+1} = \sigma_i X_i \sigma_i \quad Y_{i+1} = \sigma_i Y_i \sigma_i \quad (4.1)$$

$$X_1 Y_2 = Y_2 X_1 \sigma_1^2 \quad (4.2)$$

The results of the previous section easily imply:

**Theorem 4.3.** *There exists unique group morphisms*

$$\phi : B_n(T^2 \setminus \text{disc}) \longrightarrow (E_H^{(k)} \otimes H^{\otimes n})^\times \rtimes S_n$$

given by

$$X_1 \longmapsto X^{0,1}, \quad Y_1 \longmapsto Y^{0,1}, \quad \sigma_i \longmapsto (i, i+1) \mathcal{R}^{i,i+1}.$$

*Proof.* The two first set of cross relations can obviously be taken as a definition of  $X_i, Y_i$  for  $i > 1$ . That these operators pairwise commute follows from the reflection equation and the Yang-Baxter equation. The last cross relation is nothing but the defining equation (1.2) of  $E_H^{(k)}$ .  $\square$

Let  $\widetilde{SL_2(\mathbb{Z})}$  denote the group generated by  $A, B, Z$  with relations:

$$A^4 = (AB)^3 = Z, \quad (A^2, B) = 1. \quad (4.3)$$

Clearly,  $Z$  is central, so this is a central extension,

$$1 \rightarrow \mathbb{Z} \rightarrow \widetilde{SL_2(\mathbb{Z})} \rightarrow SL_2(\mathbb{Z}) \rightarrow 1.$$

**Proposition 4.4.** *The group  $\widetilde{SL_2(\mathbb{Z})}$  acts on  $B_n(T^2 \setminus \text{disc})$  in the following way:*

$$\begin{aligned} A \cdot \sigma_i &= \sigma_i & B \cdot \sigma_i &= \sigma_i \\ A \cdot X_1 &= Y_1 & A \cdot Y_1 &= Y_1 X_1^{-1} Y_1^{-1} \\ B \cdot X_1 &= X_1 & B \cdot Y_1 &= Y_1 X_1^{-1}. \end{aligned}$$

**Proposition 4.5.** *Let  $B$  be an algebra and  $(X_B, Y_B) \in B \otimes H$  satisfying equation (1.2) and axioms (2.1) with  $k = 1$ . Then, so does  $(X_B, Y_B X_B^{-1})$  and  $(Y, Y_B X_B^{-1} Y_B^{-1})$ .*

*Proof.* Equation (1.2) is exactly one of the defining relation of  $B_{1,n}^1$  so that it is satisfied follows from the previous proposition. So we just have to check that  $Y_B X_B^{-1}$  and  $Y_B X_B^{-1} Y_B^{-1}$  satisfies (2.1) with  $k = 1$ . This is a direct computation:

$$\begin{aligned} (Y_B X_B^{-1})^{0,12} &= \mathcal{R}^{2,1} Y_B^{0,2} \mathcal{R}^{1,2} Y_B^{0,1} (X_B^{0,1})^{-1} (\mathcal{R}^{1,2})^{-1} (X_B^{0,2})^{-1} (\mathcal{R}^{2,1})^{-1} \\ &= \mathcal{R}^{2,1} Y_B^{0,2} \mathcal{R}^{1,2} Y_B^{0,1} (\mathcal{R}^{1,2})^{-1} (X_B^{0,2})^{-1} (\mathcal{R}^{2,1})^{-1} (X_B^{0,1})^{-1} \mathcal{R}^{2,1} (\mathcal{R}^{2,1})^{-1} \\ &= \mathcal{R}^{2,1} Y_B^{0,2} \mathcal{R}^{1,2} (\mathcal{R}^{1,2})^{-1} (X_B^{0,2})^{-1} \mathcal{R}^{1,2} Y_B^{0,1} \mathcal{R}^{2,1} (\mathcal{R}^{2,1})^{-1} (X_B^{0,1})^{-1} \\ &= \mathcal{R}^{2,1} Y_B^{0,2} (X_B^{0,2})^{-1} \mathcal{R}^{1,2} Y_B^{0,1} (X_B^{0,1})^{-1}, \end{aligned}$$

where at lines 2 and 3 we use the reflection equation and the elliptic commutation relation respectively. The second part is proved by doing the exact same computation replacing  $Y_B$  by  $Y_B X_B^{-1}$  and  $X_B$  by  $Y_B$ .  $\square$

**Corollary 4.6.** *There is an action of  $\widetilde{SL_2(\mathbb{Z})}$  on  $E_H^{(1)}$ , uniquely determined by its action on canonical elements  $X, Y$  as follows:*

$$\begin{aligned} A \cdot X &= Y, & A \cdot Y &= YX^{-1}Y^{-1}, \\ B \cdot X &= X, & B \cdot Y &= YX^{-1}. \end{aligned}$$

Moreover, the action is compatible with the  $\widetilde{SL_2(\mathbb{Z})}$ -action on  $B_n(T^2 \setminus \text{disc})$ ,

*Proof.* It follows from Proposition 4.5 together with the universal property stated in Corollary 3.5.  $\square$

## 5. RELATION WITH THE HEISENBERG DOUBLE AND QUANTUM FOURIER TRANSFORM

Since  $\tilde{H}_0$  is a  $H^{[2],coop}$ -module algebra, one can form the semi-direct product  $\tilde{H} \rtimes H^{[2],coop}$ . It is easily checked that  $H \otimes 1 \subset H^{[2],coop}$  is a coideal subalgebra, hence the following definition makes sense:

**Definition 5.1.** *The Heisenberg double  $D_H$  is the subalgebra  $\tilde{H}_0 \rtimes (H \otimes 1)$ .*

*Remark 5.2.* The standard definition of the Heisenberg double involves  $H^e$  and the usual dual, instead of  $H^{[2]}$  and the braided dual. However, it is shown in [VV] that these two algebras are isomorphic.

Clearly, the double braiding  $\mathcal{R}^{2,1}\mathcal{R}^{1,2}$  satisfies axiom (2.1) with  $k = 0$ . This is a manifestation of the embedding of the cylinder braid group on  $n$  strands into the ordinary braid group on  $n + 1$  strands. We have:

**Theorem 5.3.** [Jo] *The canonical element  $X \in D_H \otimes H$  together with the image of the double braiding under the inclusion  $H \otimes H \rightarrow D_H \otimes H$  satisfy the commutation relation (1.2).*

**Corollary 5.4.** *There exists a canonical map from the elliptic double to the Heisenberg double.*

By construction, this map is the identity on the first  $\tilde{H}_0$  component and defined on the second component by the factorization map,

$$\begin{aligned} \phi : \tilde{H}_0 &\rightarrow H, \\ f &\mapsto (f \otimes id)(\mathcal{R}^{2,1}\mathcal{R}^{1,2}). \end{aligned}$$

**Definition 5.5.** *A quasi-triangular Hopf algebra is called factorizable if  $\phi$  is injective.*

Let  $I_H$  be the image of  $\phi$  and let  $D'_H$  be the subalgebra  $\tilde{H} \rtimes (I_H \otimes 1)$  of  $D_H$ .

**Theorem 5.6.** *If  $H$  is a factorizable Hopf algebra, then  $D'_H$  is isomorphic as an algebra to  $E_H^{(0)}$ .*

Let  $G$  be a reductive algebraic group,  $\mathfrak{g}$  its Lie algebra and  $U = U_q(\mathfrak{g})$  the corresponding quantum group. Recall (see e.g. [CP, Chap. 9]) that this is a quasi-triangular Hopf algebra<sup>1</sup> over  $\mathbb{C}(q)$  for  $q$  a variable which, roughly, specialize to the enveloping algebra of  $\mathfrak{g}$  at  $q = 1$ . Denote by  $U' = U_q(\mathfrak{g})'$  its ad-locally finite part.

**Theorem 5.7** ([BS, RSTS]).  *$U$  is a factorizable ribbon Hopf algebra, and the image of the factorization map  $(U^*) \rightarrow U$  is  $U'$ .*

<sup>1</sup>This is not quite true since the R-matrix does not belongs to  $U_q(\mathfrak{g})^{\otimes 2}$  but only to a certain completion of it, but it is still enough for our purpose



Let  $D_q(G)$  be the subalgebra  $\tilde{U} \rtimes U'$  of the Heisenberg double of  $U$ . It is a deformation of the algebra of differential operators on  $G$ . Thanks to the above theorem,  $D_q(G)$  is isomorphic to  $E_U^{(0)}$  which is itself isomorphic to  $E_U^{(1)}$ . Altogether this yields the action of  $\widetilde{SL_2(\mathbb{Z})}$  on  $D_q(G)$ .

## 6. RELATION TO CLASSICAL FOURIER TRANSFORM

In this section we show how the Weyl algebra of  $\mathfrak{g}$  and the classical Fourier transform can be obtained both directly as the elliptic double of a certain Hopf algebra and via an appropriate degeneration of the elliptic double of the corresponding quantum group. Let  $U_\hbar(\mathfrak{g})$  be the “formal” version of the quantum group. This is a topological quasi-triangular Hopf algebra over  $\mathbb{C}[[\hbar]]$ , where  $\hbar$  is a formal variable, deforming the enveloping algebra of  $\mathfrak{g}$  and whose definition can be found, e.g., in [CP, Chap. 6]. Since directly taking the classical (i.e.  $\hbar = 0$ ) limit of the elliptic commutation relation gives the commutative algebra  $S(\mathfrak{g})^{\otimes 2}$  we will have to consider a slightly more complicated degeneration.

Let  $S(\mathfrak{g})$  denote the symmetric algebra on  $\mathfrak{g}$ , equipped with its standard co-product  $\Delta(X) = X \otimes 1 + 1 \otimes X$  for  $X \in \mathfrak{g}$ , making it a commutative, cocommutative Hopf algebra. Let  $r \in \mathfrak{g}^{\otimes 2}$  denote the quasi-classical limit of the R-matrix of  $U_\hbar(\mathfrak{g})$ , i.e.:

$$\mathcal{R} = 1 + \hbar r + O(\hbar^2).$$

Then, in a straightforward way, the completion of the symmetric algebra  $(\widehat{S}(\mathfrak{g}), \mathcal{R}_0 = \exp(r))$  is a quasi-triangular, factorizable Hopf algebra<sup>2</sup>. Let  $t = r + r^{2,1} \in S^2(\mathfrak{g})^{\mathfrak{g}}$  and let  $C$  denote the corresponding Casimir element, i.e.  $C = m(t)$  where  $m$  is the multiplication of  $S(\mathfrak{g})$ . Then  $\nu_0 = \exp(-C/2)$  is a ribbon element. Since  $\mathcal{R}_0 \notin S(\mathfrak{g})^{\otimes 2}$ ,  $S(\mathfrak{g})$  is not strictly speaking a ribbon Hopf algebra, but the construction of the elliptic double is still well defined in this situation.

Let  $D(\mathfrak{g})$  be the algebra of differential operators on  $\mathfrak{g}$ , i.e. the Weyl algebra. As a vector space it is  $S(\mathfrak{g}^*)^{\otimes 2}$ , the two copies of  $S(\mathfrak{g}^*)$  are subalgebras and the cross relations are:

$$\forall f, g \in \mathfrak{g}^*, [f \otimes 1, 1 \otimes g] = \langle f, g \rangle$$

where  $\langle \cdot, \cdot \rangle$  is the pairing on  $\mathfrak{g}^*$  induced by  $t$ . The first result of this section is:

**Proposition 6.1.** *The 0th elliptic double of  $(S(\mathfrak{g}), \mathcal{R}_0)$  is isomorphic to the Weyl algebra  $D(\mathfrak{g})$  and the action of the generator  $A$  of  $\widetilde{SL_2(\mathbb{Z})}$  coincides with the classical Fourier transform. That is, on generators  $(f, g) \in \mathfrak{g}^* \times \mathfrak{g}^* \subset D(\mathfrak{g})$ , we have,*

$$A(f, g) = (-g, f).$$

*Proof.* Let  $x, y$  denote two copies of the canonical element in  $\mathfrak{g}^* \otimes \mathfrak{g}$ . The restricted dual of  $S(\mathfrak{g})$  is  $S(\mathfrak{g}^*)$  and the corresponding canonical element is  $X = \exp(x)$ . Since  $S(\mathfrak{g})$  is commutative, equation (2.1) reduces to the standard relation,

$$(\text{id} \otimes \Delta)(X) = X^{0,1} X^{0,2},$$

hence the braided dual and the restricted dual coincide. Likewise, the defining equation of the elliptic double reduces to:

$$(X^{0,1}, Y^{0,2}) = \mathcal{R}_0^{2,1} \mathcal{R}_0^{1,2},$$

where  $(a, b) = aba^{-1}b^{-1}$  and  $Y = \exp(y)$ . Since

$$[x^{0,1}, t^{1,2}] = [y^{0,2}, t^{1,2}] = 0,$$

this equation is equivalent to:

$$[x^{0,1}, y^{0,2}] = t^{1,2}.$$

<sup>2</sup>Here the tensor product is the topological one, i.e.  $\widehat{S}(\mathfrak{g})^{\otimes 2} := \widehat{S}(\mathfrak{g} \times \mathfrak{g})$

Applying  $f$  and  $g$  to the first and second components, respectively, of the above equation gives the defining relations (6) of  $D(\mathfrak{g})$ .

Since  $(S(\mathfrak{g}), \mathcal{R}_0)$  is ribbon,  $E_{S(\mathfrak{g})}^{(0)}$  is isomorphic to  $E_{S(\mathfrak{g})}^{(1)}$ . Pulling back the action of the  $A$  generator of  $\widetilde{SL_2(\mathbb{Z})}$  through this isomorphism, we find:

$$x \mapsto y \qquad y \mapsto Y^{-1}(-x + (1 \otimes C))Y$$

It is easily seen that the cross relations of  $D(\mathfrak{g})$  implies

$$Y^{-1}xY = x + (1 \otimes C).$$

Hence  $A$  map  $x$  to  $y$  and  $y$  to  $-x$ .  $\square$

Let  $U_{\hbar^2}(\mathfrak{g})$  be the  $\mathbb{C}[[\hbar]]$ -Hopf algebra obtained by formally replacing  $\hbar$  by  $\hbar^2$  in the definition of the product, the coproduct and the R-matrix of  $U_{\hbar}(\mathfrak{g})$ . Denote by  $\delta_n$  the map  $(\text{id} - \epsilon)^{\otimes n} \circ \Delta^n$  where  $\epsilon$  is the counit of  $U_{\hbar^2}(\mathfrak{g})$ . Denote by  $\widehat{U}$  the quantum formal series Hopf algebra (QFSHA) attached to  $U_{\hbar^2}(\mathfrak{g})$ , i.e. the sub-algebra

$$\widehat{U} = \{x \in U_{\hbar^2}(\mathfrak{g}), \delta_n(x) \in \hbar^n U_{\hbar^2}(\mathfrak{g}), \forall n \geq 0\}$$

It is known [Dr1, Ga] that  $\widehat{U}$  is a flat deformation of  $\widehat{S}(\mathfrak{g})$ . Hence, choose a  $\mathbb{C}[[\hbar]]$ -module identification

$$\psi : \widehat{U} \longrightarrow \widehat{S}(\mathfrak{g})[[\hbar]]$$

which is the identity modulo  $\hbar$ , and let  $U \subset \widehat{U}$  be the preimage under  $\psi$  of  $S(\mathfrak{g})[[\hbar]]$ .

**Proposition 6.2.** *The following holds:*

- (a)  $U$  is a Hopf algebra.
- (b) We have canonical bialgebra isomorphisms:

$$\widehat{U}/(\hbar) \cong \widehat{S}(\mathfrak{g}), \qquad U/(\hbar) \cong S(\mathfrak{g}).$$

- (c) The R-matrix of  $U_{\hbar^2}(\mathfrak{g})$  belongs to  $\widehat{U}^{\otimes 2}$  and its image in  $\widehat{S}(\mathfrak{g})^{\otimes 2}$  is  $\mathcal{R}_0$ .

One can therefore consider the 0th elliptic double of  $U$ . A direct consequence of the above proposition is then:

**Corollary 6.3.** *The algebra  $E_U$  is a flat deformation of the Weyl algebra  $D(\mathfrak{g})$ , and the  $\widetilde{SL_2(\mathbb{Z})}$ -action on  $E_U$  degenerates to the  $\widetilde{SL_2(\mathbb{Z})}$ -action on  $D(\mathfrak{g})$ . In particular, the quantum Fourier transform degenerates to the classical one.*

*Proof of Prop. 6.2.* All of this can be checked explicitly. A more conceptual argument is as follows: recall that  $(\mathfrak{g}, \mu, \delta, r)$  is a quasi-triangular Lie bialgebra, where we denote by  $\mu$  its bracket and by  $\delta$  its co-bracket. The quantum group  $U_{\hbar^2}(\mathfrak{g})$  is obtained by applying an Etingof–Kazhdan quantization functor [EK] to the  $\mathbb{C}[[\hbar]]$ -quasi-triangular Lie bialgebra  $(\mathfrak{g}[[\hbar]], \mu, \hbar^2 \delta, \hbar^2 r)$ . On the other hand,  $\widehat{U}$  is the quasi-triangular Hopf algebra obtained by applying the same functor to the quasi-triangular Lie bialgebra  $(\mathfrak{g}[[\hbar]], \hbar \mu, \hbar \delta, r)$ . The QFSHA construction is the lift of the inclusion,

$$(\mathfrak{g}[[\hbar]], \hbar \mu, \hbar \delta, r) \longrightarrow (\mathfrak{g}[[\hbar]], \mu, \hbar^2 \delta, \hbar^2 r),$$

given by  $x \mapsto \hbar x$  (since  $r \in \mathfrak{g}^{\otimes 2}$ , its image is indeed  $\hbar^2 r$ ).

One can show that the product, the coproduct and the antipode on  $\widehat{U}$  restrict to a well-defined Hopf algebra structure on  $U$ . By construction, the reduction modulo  $\hbar$  of  $\widehat{U}$  is the quantization of the  $\mathbb{C}$ -quasi-triangular Lie bialgebra,

$$(\mathfrak{g}[[\hbar]], \hbar \mu, \hbar \delta, r)/(\hbar) \cong (\mathfrak{g}, 0, 0, r),$$

which is easily seen to be  $(\widehat{S}(\mathfrak{g}), \mathcal{R}_0)$ .  $\square$

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